Properties of Conjugate of a Complex Numbers

Properties of $\overline{z}$

$If\ z = x + iy$ is a complex number then the complex conjugate $\overline{z}$ of z is defined as

$\overline{z}$= $x - iy$

1 $\overline{\overline{z}}$ = z

e.g. if $z = 5 + 2i$

then $\overline{z}$ = $5 - 2i$

$\overline{\overline{z}}$ = $5 + 2i$ = z

2 $\overline{z_{1+}z_2}$ = $\overline{z_1}$+$\overline{z_2}$

Let z_1=$3 + 2i$ and z_2= $5 - 3i$

Then $\overline{z_1 + z_2}$ = $\overline{(3 + 2i) + (5 - 3i)}$

= $\overline{8 - i}$

= $8 + i$ ……………………….. (1)

Now $\overline{z_1} + \overline{z_2}$ = $\overline{(3 + 2i)} + \overline{(5 - 3i)}$

= $3 - 2i + 5 + 3i$

$= 8 + i$ ....(2)

From (1) and (2) verified

3 $\overline{z_1 z_2} = \overline{z_1}.\overline{z_2}$

$\overline{z_1 z_2} = \overline{(3+2i)(5-3i)}$

$= \overline{21+i}$

$= 21 - i$ ....(1)

$\overline{z_1}\,\overline{z_2} = \overline{(3+2i)}\,\overline{(5-3i)}$

$= (3-2i)(5+3i)$

$= 21 - i$ ....(2)

From (1) and (2) its verified

4 $\overline{z^{-1}} = (\overline{z})^{-1}$

If $z = 5 + 2i$ then $z^{-1} = \frac{5}{29} - \frac{2}{29}i$

$\overline{z^{-1}} = \frac{5}{29} + \frac{2}{29}i$

Now $\overline{z} = 5 - 2i$ and $(\overline{z})^{-1} = \frac{5}{29} + \frac{2}{29}i$ verified

5 Always

$|z| = |\overline{z}|$ and $\text{Re}(z) \leq |z|$

- If $z_{1\ and}\ z_2$ are two complex numbers then prove that

$$|z_1 + z_2| \leq |z_1| + |z_2|$$

Proof – We know that

$|z_1 + z_2|^2 = (z_1 + z_2)\,(\overline{z_1 + z_2})$

$= (z_1 + z_2)\,(\overline{z_1} + \overline{z_2})$

$= z_1\overline{z_1} + z_1\overline{z_2} + z_2\overline{z_1} + z_2\overline{z_2}$

$= |z_1|^2 + 2\text{Re}\,(z_1\overline{z_2}) + |z_2|^2$

$\leq |z_1|^2 + 2|z_1||z_2| + |z_2|^2$

$\leq (|z_1| + |z_2|)^2$

Thus, we have

$|z_1 + z_2|^2 \leq (|z_1| + |z_2|)^2$

$\therefore |z_1 + z_2| \leq |z_1| + |z_2|$

Hence proved.

- Prove that, for any two complex numbers $z_{1\ and}\ z_2$

$$\big||z_1| - |z_2|\big| \leq |z_1 + z_2|$$

First write z_1 as

$$z_1 = z_1 + z_2 + (-z_2)$$

$|z_1| = |(z_1 + z_2) + (-z_2)|$

$|z_1| \leq |z_1 + z_2| + |-z_2|$

$|z_1| \leq |z_1 + z_2| + |-z_2|$

$|z_1| \leq |z_1 + z_2| + |z_2|$

$|z_1| - |z_2| \leq |z_1 + z_2|$…………………(1)

$Let \quad z_2 = z_2 + z_{1+}(-z_1)$

$\therefore |z_2| = |z_2 + z_1 + (-z_1)|$

$\therefore |z_2| \leq |z_2 + z_1| + |-z_1|$

$\therefore |z_2| \leq |z_2 + z_1| + |z_1|$

$\therefore |z_2|$ - $|z_1| \leq |z_2 + z_1|$

$\therefore -|z_2|$ + $|z_1| \geq -|z_1 + z_2|$

$\therefore -|z_1 + z_2| \leq -|z_2|$ + $|z_1|$……………………..(2)

From 1 and 2 we get

$-|z_1 + z_2| \leq |z_1| - |z_2| \leq |z_1 + z_2|$

Gives us

$\big||z_1| - |z_2|\big| \leq |z_1 + z_2|$

Hence proved

- Prove that for all complex numbers $z_{1\ and}\ z_2$

$|z_1 + z_2|^2 + |z_1 - z_2|^2$= 2$|z_1|^2$ + 2$|z_2|^2$

Solution we know that

= $|z_1 + z_2|^2$= $(z_1 + z_2)(\overline{z_1 + z_2})$

= $(z_1 + z_2)$ $(\overline{z_1}$+ $\overline{z_2})$

= $z_1\overline{z_1}$ + $z_1\overline{z_2}$+ $z_2\overline{z_1}$+ $z_2\overline{z_2}$

= $|z_1|^2$+ $z_1\overline{z_2}$+ $z_2\overline{z_1}$+ $|z_2|^2$……………………….(1)

Similarly

$|z_1 - z_2|^2$ = $(z_1 - z_2)(\overline{z_1 - z_2})$

= $(z_1 - z_2)$ $(\overline{z_1}$- $\overline{z_2})$

= $z_1\overline{z_1}$ - $z_1\overline{z_2}$- $z_2\overline{z_1}$+ $z_2\overline{z_2}$

$= |z_1|^2 - z_1\overline{z_2} - z_2\overline{z_1} + |z_2|^2$ ……………………….(2)

Adding (1) and (2), we get

$$|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2|z_1|^2 + 2|z_2|^2$$

Hence proved

- ❖ For any complex number $z = x + iy$, prove that $|z| \leq |Re(z)| + |Im(z)|$

  To prove this, we will show that

  $\sqrt{x^2 + y^2} \leq |x| + |y|$

  For this, consider

  $(|x| + |y|)^2 = |x|^2 + |y|^2 + 2|x||y|$

$$i.e. \quad (|x| + |y|)^2 \geq |x|^2 + |y|^2$$

$i.e. \quad (|x| + |y|)^2 \geq x^2 + y^2$

Taking square root both sides we get

$$(|x| + |y|) \geq \sqrt{x^2 + y^2}$$

$$i.e. \quad |x| + |y| \geq |z|$$

$$i.e. \quad |z| \leq |x| + |y|$$

$$i.e. \quad |z| \leq |Re(z)| + |Im(z)|$$

Hence proved.

THANK YOU